Controllability of open quantum systems with Kraus-map dynamics

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Abstract

This paper presents a constructive proof of complete kinematic state controllability of finite-dimensional open quantum systems whose dynamics are represented by Kraus maps. For any pair of states (pure or mixed) on the Hilbert space of the system, we explicitly show how to construct a Kraus map that transforms one state into another. Moreover, we prove by construction the existence of a Kraus map that transforms *all* initial states into a predefined target state (such a process may be used, for example, in quantum information dilution). Thus, in sharp contrast to unitary control, Kraus-map dynamics allows for the design of controls which are robust to variations in the initial state of the system. The capabilities of non-unitary control for population transfer between pure states illustrated for an example of a two-level system by constructing a family of non-unitary Kraus maps to transform one pure state into another. The problem of dynamic state controllability of open quantum systems (i.e., controllability of state-to-state transformations, given a set of available dynamical resources such as coherent controls, incoherent interactions with the environment, and measurements) is also discussed.

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1. Introduction

Coherent control of quantum systems is a rapidly developing area of research with applications to numerous physical and chemical problems [1–5]. The general goal of quantum control is to manipulate the dynamics of a quantum system in a desired way by applying suitable external control fields, typically, optimally shaped pulses of a coherent radiation field. Much theoretical work [6–8] has been devoted to coherent control of closed quantum systems with unitary dynamics. However, realistic physical situations entail control of open quantum systems whose dynamics is non-unitary due to interactions with the environment. Research

on various aspects of control of open quantum systems has appeared in recent years [9–20], motivated by many applications including quantum computing [21–24], laser cooling [25–28], quantum reservoir engineering [29], management of decoherence [30–40], chemical reactions and energy transfer in molecules [41–45].

A coherent control field acts on the system through the Hamiltonian part of its dynamics. A qualitatively different approach relies on using specially tailored environments, which affect the system via non-unitary evolution, with controls applied through the dissipative part of the dynamics [46]. In this approach, a suitably optimized non-equilibrium distribution function of an environment (e.g., an electron, atom, or molecular gas, or a solvent) is employed as a control instrument to achieve the desired objective. This type of incoherent control by the environment (ICE) may be combined with optimally tailored coherent fields to allow for simultaneous control through both the Hamiltonian and dissipative parts of the system dynamics.

One of the fundamental issues of quantum control assesses the system's controllability. A quantum system is controllable in a set of configurations, $\mathcal{S} = \{\lambda\}$, if for any pair of configurations $\lambda_1, \lambda_2 \in \mathcal{S}$ there exists a time-dependent control, $c(t)$, that can drive the system from the initial configuration λ_1 to the final configuration λ_2 in a finite time T. Here, the notion of configuration means either the state of the system ρ, the expectation value of an observable $\mathrm{Tr}(\rho O)$, the evolution operator $U(t)$, or the Kraus map Φ, depending on the specific control problem. Controllability of closed quantum systems with unitary dynamics has been extensively studied [47–58]. We will briefly review some of these relevant results in section 2.

Unitary dynamics can achieve control only within sets of states exhibiting the same density-matrix spectrum, and a unitary transformation cannot connect two quantum states of different purity. Non-unitary evolution of open quantum systems is able to lift this restriction and transform pure states into mixed ones and vice versa (a familiar example is the cooling of a thermalized quantum system, which requires coupling to a reservoir). However, the important question of controllability of open quantum systems is not yet fully addressed, although some aspects of this problem have been considered. In particular, controllability of a quantum system undergoing non-unitary evolution and controlled by a coherent field, which acts only through the Hamiltonian part of the dynamics, has been discussed in a number of works [11–13]. Another related research direction concerns supplementing unitary coherent controls by measurements [10, 15–17, 19, 20].

In this paper, we take a different perspective by considering the problem of kinematic state controllability (KSC) of open quantum systems whose dynamics are represented by Kraus maps [59, 60]. Specifically, we prove the existence of a Kraus map that can move a finite-dimensional open quantum system from any initial state ρ_{in} to any final target state ρ_{f}. This establishes complete KSC of finite-dimensional open quantum systems with Kraus-map dynamics, in contrast to restricted KSC of closed quantum systems where unitary dynamics can connect only states with the same density-matrix spectrum.

The constructed Kraus map transforms *all* initial states into a predefined final target state ρ_{f}. Such Kraus transformations can be used, for example, in the context of quantum information dilution [61, 62] to realize a mapping of an unknown (mixed or pure) quantum state into a given target state. The existence of such all-to-one maps is a significant distinction between non-unitary evolution and unitary evolution, since in the latter case the evolution operator and the corresponding coherent control field always depend on both the initial and target states of the system. Therefore, extending the controls to include appropriate non-unitary dynamics allows for solving the problem of achieving control operations which are robust to variations in the initial state of the system. Robustness to variations in the initial state

is understood here as the ability to use a single control (i.e., a single Kraus map) to transfer all initial states into a predefined, mixed or pure, final target state. The possibility of using a single Kraus map to transform all initial states into a given final state is a property stronger than transitivity of the set of Kraus maps on the set of density matrices in a finite-dimensional Hilbert space. This property does not have an analogue for unitary transformations. While the set of unitary operators acts transitively on the set of unit-norm vectors, no single unitary transformation can map all pure states into a given final pure state.

In practice, the unitary or non-unitary dynamics of the system is guided by a set of available controls. Possible controls include pulses of coherent electromagnetic radiation, incoherent environments (e.g., electron, atom, or molecular gases, or a solvent) with tunable non-equilibrium distribution functions, disturbances induced by quantum measurements, etc. The ability to make transformations between the states of the system, using the available set of controls, is referred to as dynamic state controllability (DSC). For a specific problem, DSC is determined by the particular dependence of the Kraus operators on the controls. In this paper, we discuss some general properties of DSC which do not require knowledge of this dependence.

Several important problems still remain open, including a study of DSC for specific quantum systems, taking into account the available laboratory control tools, and an analysis of robustness of the dynamical control to imperfections and environmental couplings during the evolution. Some attempts in this direction exist, including, for example, an analysis of coherent control of non-dispersive wave packets [63], although the problem remains generally open for a future study. However, first it would be desirable to lay the ground for these system-dependent studies by performing a general analysis of KSC and DSC, which can reveal the highest degree of control attained by general physically allowable dynamics. The existence of quantum controls which are robust to variations in the initial states, established in the present work, should facilitate an exploration of non-unitary control tools for specific systems.

Experimental studies of environmentally-induced decoherence in open quantum systems explored various physical processes, including the loss of spatial coherence of an atomic wavefunction due to spontaneous emission [64], decoherence of motional superposition states of a trapped atom coupled to engineered reservoirs [65], the loss of spatial coherence in matter-wave interferometry with fullerenes caused by collisions with an environmental gas [66] or by thermal emission of radiation [67], and decoherence in networks of spin qubits due to pairwise dipolar interactions between the spins [68]. Control of decoherence was experimentally explored in several systems, including photon pairs generated from atomic ensembles [38], nuclear spin qubits in fullerenes [39] and vibrational wave-packets in diatomic molecules [40]. Suppression of decoherence for quantum computing using environment induced quantum Zeno effect was suggested [69]. Control of decoherence of motional quantum states caused by scattering events was proposed in [70]. The feasibility of decoherence suppression via manipulations of quantum states is evidently related to a more general problem of state controllability. For example, the controllability analysis can be applied to examine the feasibility of transforming an arbitrary initial state into a state in a weak-decoherence subspace. Considering the rapidly growing interest in experimental management of decoherence [38–40], our theoretical analysis hopefully will stimulate further laboratory studies of practical aspects of state controllability in open quantum systems.

The paper is organized as follows. In section 2, we formulate definitions of controllability for closed and open quantum systems. The general proof of open-system KSC is presented in section 3. The capabilities of non-unitary control for population transfer between pure states are illustrated in section 4 for two-level quantum systems. In section 5, we also discuss general conditions for DSC of open quantum systems undergoing Kraus-map evolution.

2. Formulation of the controllability analysis

2.1. Definitions of controllability for closed quantum systems

A discussion of different notions of controllability in finite-dimensional closed quantum systems with unitary dynamics is available [54, 55]. Here, we briefly review some basic definitions and results as background for consideration of the open-system controllability analysis to follow.

For unitary dynamics, KSC is defined as follows:

Definition 1. *A closed quantum system with unitary dynamics is kinematically controllable in a set S_K of states if for any pair of states $\rho_1 \in S_K$ and $\rho_2 \in S_K$ there exists a unitary operator U, such that $\rho_2 = U\rho_1 U^\dagger$.*

Any two quantum states that belong to the same kinematically controllable set S_K are called kinematically equivalent. It is straightforward to see [54] that two states ρ_1 and ρ_2 of a closed quantum system are kinematically equivalent if and only if they have the same eigenvalues. Therefore, all quantum states that belong to the same kinematically controllable set have the same density-matrix eigenvalues, the same von Neumann entropy and the same purity. For example, all pure states belong to the same kinematically controllable set. However, any pure state is not kinematically equivalent to any mixed state. For a closed quantum system all states on the system's Hilbert space are separated into unconnected sets of kinematically equivalent states.

The dynamics of a closed quantum system is governed by the Schrödinger equation:

$$i\hbar \frac{dU(t)}{dt} = HU(t), \qquad U(0) = I. \tag{1}$$

Here, H is the Hamiltonian, $U(t)$ is the evolution operator and I is the identity operator. Assuming that the Hamiltonian H is a functional of a set of time-dependent controls $H = H[c_1(t), \ldots, c_k(t)]$, DSC for unitary evolution is defined as follows:

Definition 2. *A closed quantum system with unitary evolution is dynamically controllable in a set S_D of states if for any pair of states $\rho_1 \in S_D$ and $\rho_2 \in S_D$ there exist a finite time T and a set of controls $\{c_1(t), \ldots, c_k(t)\}$, such that the solution $U(T)$ of the Schrödinger equation (1) transforms ρ_1 into ρ_2: $\rho_2 = U(T)\rho_1 U^\dagger(T)$.*

Since unitary dynamics can be controlled only within the set of kinematically equivalent states, a dynamically controllable set of states S_D is always a subset of the corresponding kinematically controllable set S_K. If the dynamically controllable set of pure states coincides with its kinematically controllable counterpart (i.e., the set of all pure states), the closed quantum system is called pure-state controllable. If all dynamically controllable sets of states coincide with their kinematically controllable counterparts, the system is called density-matrix controllable.

It is possible to define controllability of a closed quantum system not only in a set of states, but also in a set of evolution operators $U(t)$. The corresponding property, called evolution-operator controllability (EOC), is defined as follows:

Definition 3. *A closed quantum system with unitary dynamics is evolution-operator controllable if for any unitary operator V there exists a finite time T and a set of controls $\{c_1(t), \ldots, c_k(t)\}$, such that $V = U(T)$, where $U(T)$ is the solution of the Schrödinger equation (1) with $H = H[c_1(t), \ldots, c_k(t)]$.*

For an N-level closed system, a necessary and sufficient condition for EOC is [54, 55] that the dynamical Lie group $\mathfrak{G}$ of the system be U(N) (or SU(N) for a traceless Hamiltonian, which differs from the original one just by a physically irrelevant shift in the energy). It can be also shown [54, 55] that EOC is equivalent to density-matrix controllability, while the condition for pure-state controllability is weaker.

2.2. Definition of KSC for open quantum systems

The state of an open quantum system is represented by the reduced density matrix $\rho = \mathrm{Tr_E}(\rho_{\mathrm{tot}})$, where ρ_{tot} is the density matrix of the system and environment taken together, and $\mathrm{Tr_E}$ denotes the trace over the environment degrees of freedom. The dynamics of open quantum systems is governed by various master equations (see, e.g., [71] on master equations for a system weakly interacting with the environment and recent works on master equations for collisional decoherence with a strong interaction [72–75]). If the system and environment are initially uncorrelated, the time evolution of the system in the kinematic picture can be described by a completely positive, trace-preserving linear map.

Let $\mathcal{H}$ be the Hilbert space of the system and $\mathcal{T}(\mathcal{H})$ be the space of trace-class operators on $\mathcal{H}$. For example, for an N-level quantum system, $\mathcal{H} = \mathbb{C}^N$ is the space of complex vectors of length N and $\mathcal{T}(\mathcal{H}) = \mathcal{M}_N$ is the space of $N \times N$ complex matrices. The set of density matrices (i.e., the set of positive operators on $\mathcal{H}$ with trace one) is denoted as $\mathcal{D}(\mathcal{H})$ (clearly, $\mathcal{D}(\mathcal{H}) \subset \mathcal{T}(\mathcal{H})$).

Definition 4. *A linear map* $\Phi : \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{T}(\mathcal{H})$ *is called completely positive if the map* $\Phi \otimes I_l : \mathcal{T}(\mathcal{H}) \otimes \mathcal{M}_l \rightarrow \mathcal{T}(\mathcal{H}) \otimes \mathcal{M}_l$ *(where I_l is the identity map in $\mathcal{M}_l$) is positive for any* $l \in \mathbb{N}$. *The map Φ is called trace preserving if for any* $\rho \in \mathcal{T}(\mathcal{H}), \mathrm{Tr}(\Phi[\rho]) = \mathrm{Tr}(\rho)$.

Any completely positive, trace-preserving map has the Kraus operator-sum representation [59, 60, 76]:

$$\Phi[\rho] = \sum_{i=1}^{n} K_i \rho K_i^{\dagger}, \tag{2}$$

where the Kraus operators K_i satisfy the condition

$$\sum_{i=1}^{n} K_i^{\dagger} K_i = I. \tag{3}$$

Here, $n \in \mathbb{N}$ is the number of the Kraus operators K_i and I is the identity operator on $\mathcal{H}$. Condition (3) ensures the preservation of the trace: $\mathrm{Tr}(\Phi[\rho]) = \mathrm{Tr}(\rho)$. In this paper, we refer to completely positive, trace-preserving maps simply as Kraus maps. Unitary transformations of the system states form a particular subset of Kraus maps corresponding to $n = 1$. Note that a composition of any two Kraus maps Φ_1 and Φ_2 is another Kraus map:

$$\Phi_2[\Phi_1[\rho]] \equiv (\Phi_2 \circ \Phi_1)[\rho] = \Phi_3[\rho]. \tag{4}$$

It is well known that any Kraus map Φ has infinitely many different Kraus operator-sum representations of the form (2). Let $\{K_1, \ldots, K_n\}$ be a set of Kraus operators representing Φ. For $m \geqslant n$, consider an $m \times n$ matrix W with elements w_{ij}, such that $W^{\dagger}W = I_n$. Define a new set of Kraus operators:

$$\tilde{K}_i = \sum_{j=1}^{n} w_{ij} K_j, \qquad i = 1, \ldots, m. \tag{5}$$

Then for any $\rho \in \mathcal{T}(\mathcal{H})$, one has

$$\sum_{i=1}^{m} \tilde{K}_i \rho \tilde{K}_i^{\dagger} = \sum_{i=1}^{n} K_i \rho K_i^{\dagger}, \tag{6}$$

i.e., both sets of Kraus operators, $\{K_1, \ldots, K_n\}$ and $\{\tilde{K}_1, \ldots, \tilde{K}_m\}$, represent the same Kraus map Φ. Moreover, if two different sets of Kraus operators represent the same Kraus map, then they are necessarily related by (5) with a matrix W such that $W^{\dagger}W = I$. Any Kraus map for an N-level quantum system can be represented by a set of $n \leqslant N^2$ Kraus operators [76]. That is, if the map is represented by a set of $n > N^2$ Kraus operators, there always exists another representation with not more than N^2 operators.

In the context of the system coupled to the environment, complete positivity of a map Φ means that for any admissible evolution of the system density matrix, $\Phi[\rho]$, the initial density matrix $\rho_{\text{tot}}(0)$ of the system and environment taken together will evolve into another density matrix [22]. If the system and environment are initially correlated, the map describing the evolution of the system density matrix will not be always completely positive and the Kraus operator-sum representation (2) will not be always valid [77, 78].

The notions of closed-system controllability presented in section 2.1 need to be modified for open quantum systems which allow non-unitary dynamics. For an open quantum system with Kraus-map dynamics, KSC is defined as follows:

Definition 5. *An open quantum system with the Kraus-map dynamics of the form (2) is kinematically controllable in a set $\mathcal{S}_{\text{K}}$ of states if for any pair of states $\rho_1 \in \mathcal{S}_{\text{K}}$ and $\rho_2 \in \mathcal{S}_{\text{K}}$ there exists a Kraus map Φ, such that $\rho_2 = \Phi[\rho_1]$.*

In the next section we will prove that KSC for a finite-dimensional open system with Kraus-map dynamics is *complete*, i.e., that the system is kinematically controllable in the set $\mathcal{S}_{\text{K}} = \mathcal{D}(\mathcal{H})$ of *all* density operators ρ on the Hilbert space $\mathcal{H}$. The problem of DSC for the Kraus-map evolution will be discussed in section 5.

3. Proof of complete KSC for open quantum systems with Kraus-map dynamics

The proof of complete KSC for open quantum systems with Kraus-map dynamics is given here by construction. We start by reformulating the known result that Kraus maps can transform all states of a finite-dimensional open quantum system into a given pure state. Then we prove that open quantum systems with Kraus map dynamics are completely kinematically controllable. Moreover, we prove by construction a stronger result—the existence of Kraus maps which transform all states of an open quantum system into an arbitrary (not necessarily pure) target state. We also construct Kraus maps which transform a given pure state into an arbitrary (mixed or pure) target state.

It is known (see, for example, section 8.3.1 of [21]) that Kraus maps can transform mixed states into pure ones. We now reformulate this result in a general way.

Theorem 1. *For any pure state $\rho_{\text{p}} = |\psi\rangle\langle\psi|$ on the Hilbert space $\mathcal{H}$ of a finite-dimensional open quantum system, there exists a Kraus map Φ_{atp}, such that $\Phi_{\text{atp}}[\rho] = \rho_{\text{p}}$ for all states ρ on $\mathcal{H}$[1].*

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[1] Abbreviation atp of *all to pure* is used to indicate that the Kraus map Φ_{atp} transforms all states into a pure state.

Proof. Define the operators $K_i^{(\text{atp})} = |\psi\rangle\langle\chi_i|$ for $i = 1, \ldots, N$, where N is the dimension of the system and $\{\chi_i\}$ is an arbitrary orthonormal basis in $\mathcal{H}$. The operators $K_i^{(\text{atp})}$ satisfy the normalization condition (3) and define the Kraus map

$$\Phi_{\text{atp}}[\rho] = \sum_{i=1}^{N} K_i^{(\text{atp})} \rho K_i^{(\text{atp})\dagger}. \tag{7}$$

This map transforms all states into the pure state ρ_{p} since for any density matrix ρ

$$\Phi_{\text{atp}}[\rho] = \sum_{i=1}^{N} |\psi\rangle\langle\chi_i|\rho|\chi_i\rangle\langle\psi| = (\text{Tr}\,\rho)|\psi\rangle\langle\psi| = \rho_{\text{p}}. \tag{8}$$

This completes the proof. $\qquad\square$

The choice of the orthonormal basis $\{\chi_i\}$ in the above proof is completely arbitrary. Different bases $\{\chi_i\}$ determine different sets of Kraus operators $\left\{K_i^{(\text{atp})}\right\}$ which are related to each other by (5) and all represent the same Kraus map Φ_{atp}. The map Φ_{atp} transforms all initial states ρ into the same final state ρ_{p}, i.e., it is an all-to-one map.

Next, we generalize the result of theorem 1 to the case of an arbitrary (not necessarily pure) target state.

Theorem 2. *For any state ρ_{f} on the Hilbert space $\mathcal{H}$ of a finite-dimensional open quantum system, there exists a Kraus map Φ such that $\Phi[\rho] = \rho_{\text{f}}$ for all states ρ on $\mathcal{H}$.*

Proof. Let the spectral decomposition of the final state ρ_{f} be

$$\rho_{\text{f}} = \sum_{i=1}^{N} p_i |\phi_i\rangle\langle\phi_i|, \tag{9}$$

where p_i is the probability to find the system in the state $|\phi_i\rangle$ ($p_i \geqslant 0$ and $\sum_{i=1}^{N} p_i = 1$). In particular, a pure state $\rho_{\text{f}} = |\phi\rangle\langle\phi|$ has only one non-zero eigenvalue, $p_1 = 1$. For an arbitrary orthonormal basis $\{\chi_j\}$ in $\mathcal{H}$, define the operators

$$K_{ij} = \sqrt{p_i} |\phi_i\rangle\langle\chi_j|, \qquad i, j = 1, \ldots, N. \tag{10}$$

The operators K_{ij} satisfy the normalization condition (3) and define the Kraus map:

$$\Phi[\rho] = \sum_{i,j=1}^{N} K_{ij} \rho K_{ij}^{\dagger}. \tag{11}$$

The map Φ acts on any state ρ on $\mathcal{H}$ as

$$\Phi[\rho] = \sum_{i,j=1}^{N} p_i |\phi_i\rangle\langle\chi_j|\rho|\chi_j\rangle\langle\phi_i| = (\text{Tr}\,\rho) \sum_{i=1}^{N} p_i |\phi_i\rangle\langle\phi_i| = \rho_{\text{f}}. \tag{12}$$

This completes the proof. $\qquad\square$

Complete KSC for open quantum systems with Kraus-map dynamics directly follows from theorem 8 and can be expressed in the form of corollaries.

Corollary 1. *For any pair of states ρ_1 and ρ_2 on the Hilbert space $\mathcal{H}$ of a finite-dimensional open quantum system, there exists a Kraus map Φ such that $\Phi[\rho_1] = \rho_2$.*

Corollary 2. *A finite-dimensional open quantum system with Kraus-map dynamics is kinematically controllable in the set $\mathcal{S}_{\text{K}} = \mathcal{D}(\mathcal{H})$ of all density operators on $\mathcal{H}$.*

Since the choice of the orthonormal basis $\{\chi_j\}$ is completely arbitrary, there exist infinitely many sets of Kraus operators of the form (10) (corresponding to different choices of $\{\chi_j\}$), all of which represent the same Kraus map Φ of (11). The Kraus map Φ transforms all initial states into the final target state ρ_f, i.e., it is an all-to-one map.

Given a pair of states ρ_{in} and ρ_f on the Hilbert space of a finite-dimensional open quantum system, there exist many different Kraus maps transforming ρ_{in} into ρ_f. Consider, for example, a pure initial state $\rho_{in} = |\psi\rangle\langle\psi|$ and an arbitrary (mixed or pure) final state $\rho_f = \sum_{i=1}^{N} p_i |\phi_i\rangle\langle\phi_i|$. Let $\mathcal{U} = \{U_i\}_{i=1}^{N}$ be a set of unitary operators such that $U_i|\psi\rangle = |\phi_i\rangle$. Define the operators

$$K_i^{(\text{pta})} = \sqrt{p_i} U_i, \qquad i = 1, \ldots, N, \tag{13}$$

which satisfy the normalization condition (3) and determine the Kraus map[2]

$$\Phi_{\text{pta}}[\rho] = \sum_{i=1}^{N} K_i^{(\text{pta})} \rho K_i^{(\text{pta})\dagger}. \tag{14}$$

The Kraus map Φ_{pta} transforms the pure state ρ_{in} into the state ρ_f:

$$\Phi_{\text{pta}}[\rho_{in}] = \sum_{i=1}^{N} p_i U_i |\psi\rangle\langle\psi| U_i^{\dagger} = \sum_{i=1}^{N} p_i |\phi_i\rangle\langle\phi_i| = \rho_f. \tag{15}$$

The choice of a set $\mathcal{U} = \{U_i\}$ of unitary operators with the property $U_i|\psi\rangle = |\phi_i\rangle$ is not unique. Given any such a set $\mathcal{U}$, the corresponding Kraus map Φ_{pta} will be denoted as $\Phi_{\text{pta}}^{(\mathcal{U})}$. Different sets $\mathcal{U} = \{U_i\}$ and $\tilde{\mathcal{U}} = \{\tilde{U}_i\}$, where $U_i|\psi\rangle = \tilde{U}_i|\psi\rangle = |\phi_i\rangle$, can produce different Kraus maps $\Phi_{\text{pta}}^{(\mathcal{U})}$ and $\Phi_{\text{pta}}^{(\tilde{\mathcal{U}})}$, respectively. All these maps satisfy (15) when they act on the particular state $\rho_{in} = |\psi\rangle\langle\psi|$, but in general $\Phi_{\text{pta}}^{(\mathcal{U})}[\rho] \neq \Phi_{\text{pta}}^{(\tilde{\mathcal{U}})}[\rho]$ if $\rho \neq \rho_{in}$. Every Φ_{pta} is a one-to-one map, i.e., in general $\Phi_{\text{pta}}[\rho] \neq \Phi_{\text{pta}}[\rho']$ if $\rho \neq \rho'$.

For a given final state ρ_f of the form (9), we find that the Kraus operators K_{ij} of (10) can be constructed as

$$K_{ij} = K_i^{(\text{pta})} K_j^{(\text{atp})} = \sqrt{p_i} U_i |\psi\rangle\langle\chi_j| = \sqrt{p_i}|\phi_i\rangle\langle\chi_j|, \tag{16}$$

for arbitrary choices of $|\psi\rangle$, $\{U_i\}$ and $\{\chi_j\}$. Therefore, the Kraus map Φ of (11), which transforms all states into a given final state ρ_f, can be constructed as the composition of the maps Φ_{atp} and Φ_{pta},

$$\Phi = \Phi_{\text{pta}} \circ \Phi_{\text{atp}}. \tag{17}$$

Indeed, using (8) and (15), we obtain

$$\Phi[\rho] = \Phi_{\text{pta}}[\Phi_{\text{atp}}[\rho]] = \Phi_{\text{pta}}[|\psi\rangle\langle\psi|] = \rho_f, \tag{18}$$

for all states ρ on $\mathcal{H}$.

Different constructions (equations (17) and (11), respectively) used for obtaining the same Kraus map Φ indicate the possibility of steering the system to the target state via different control pathways. The construction of (17) can be interpreted as a two-step process in which the system is first driven to a specific pure state (which is not necessarily the ground state) and subsequently transformed from this pure state into the target state (which can be either pure or mixed). The construction of (11) describes a transformation to the target state with no intermediate pure states involved. An example of such a process is the evolution of a system coupled to a thermal reservoir kept at the inverse temperature β. In this case, under

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[2] Abbreviation pta of *pure to any* is used to indicate that the corresponding Kraus map transforms a specific pure state into a given mixed or pure state.

some general conditions on the system-environment interaction, all initial system states will eventually evolve into the same thermal state $\rho = \mathrm{e}^{-\beta H_0}/\mathrm{Tr}(\mathrm{e}^{-\beta H_0})$, where H_0 is the free system Hamiltonian. At that, a mixed initial state will always stay mixed during this type of evolution.

4. Non-unitary transformations between pure states

In coherent control, transitions between pure states are achieved via unitary transformations. Unitary dynamics correspond to keeping only one term in the Kraus operator-sum representation (2). Here we show, using as an example a two-level open quantum system, that a multitude of *non-unitary* Kraus maps can be used for transforming one pure state into another.

Let the initial and final pure states be the ground $|0\rangle$ and excited $|1\rangle$ states of a two-level system, with density matrices $\rho_{\mathrm{in}} = |0\rangle\langle 0|$ and $\rho_{\mathrm{f}} = |1\rangle\langle 1|$, respectively. Define the two Kraus operators:

$$K_1 = x_1|1\rangle\langle 0| + x_2|0\rangle\langle 1|, \qquad K_2 = x_3|1\rangle\langle 0| + x_4|0\rangle\langle 1|, \tag{19}$$

where $x_i \in \mathbb{C}$, $|x_1|^2 + |x_3|^2 = 1$, and $|x_2|^2 + |x_4|^2 = 1$. The operators (19) satisfy the normalization condition (3) and define the Kraus map

$$\Phi_{\mathrm{ptp}}[\rho_{\mathrm{in}}] = \sum_{i=1}^{2} K_i \rho_{\mathrm{in}} K_i^{\dagger} = \rho_{\mathrm{f}}, \tag{20}$$

which transforms the initial state ρ_{in} into the final state ρ_{f}[3].

If the Kraus operators K_1 and K_2 of (19) are linearly dependent, $K_1 = zK_2$ ($z \in \mathbb{C}$), then Φ_{ptp} is a unitary map. However, if the Kraus operators K_1 and K_2 are linearly independent, the corresponding Kraus map Φ_{ptp} represents *non-unitary* evolution that steers a two-level open quantum system between two pure states. In this case, the choice of the parameters x_i in (19) is arbitrary up to the conditions of normalization and linear independence, and therefore there exist infinitely many pairs of Kraus operators $\{K_1, K_2\}$ which define different Kraus maps with the same property (20). Note also that Kraus maps Φ_{ptp} of (20) are one-to-one maps and differ from the all-to-one map Φ defined by (11). This emphasizes the existence not only of a multitude of different operator-sum representations of the same map, but of qualitatively different Kraus maps, all of which are capable of moving the open quantum system between the same pair of states by non-unitary dynamics.

The influence of the environment on an open quantum system is typically viewed as hindering unitary control pathways which would be otherwise effective for the closed system. However, the possibility of transforming pure states into pure states via non-unitary dynamics reveals a plethora of control pathways for open quantum systems. The existence of a multitude of non-unitary control pathways implies flexibility and possibly control robustness in the sense that if some transitions are blocked due to dynamical restrictions, other pathways may still allow the controls to move the dynamics forward. The existence of non-unitary controls, which nevertheless maintain coherence of the initial state, may be useful for quantum information applications in which the loss of coherence is a serious impediment.

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[3] Abbreviation ptp of *pure to pure* is used to indicate that the corresponding Kraus map Φ_{ptp} transforms a pure state into a pure state.

5. Conditions for dynamic state controllability of open quantum systems with Kraus-map evolution

An important question yet to be fully resolved is DSC of open quantum systems. In order to study the problem of DSC one needs to specify the dynamical capabilities, i.e., the set of available controls. While for a closed quantum system with unitary dynamics all available controls are coherent, the Kraus-map dynamics of an open system can be induced by both coherent and incoherent controls (the former act only through the Hamiltonian part of the dynamics, while the latter include interactions with other quantum systems and measurements). Let $\mathcal{C}$ be a set of all available finite-time controls, which may include coherent electromagnetic fields, tunable distribution functions of various environments, measurements, etc. Each particular configuration of controls, $c(t) \in \mathcal{C}$, induces the corresponding time evolution of the system through the Kraus map $\Phi_{c,t}$ that transforms an initial state $\rho(0)$ into the state $\rho(t) = \Phi_{c,t}[\rho(0)]$ at time t. Based on these considerations, we introduce the following definition of open-system DSC:

Definition 6. *An open quantum system with Kraus-map evolution is dynamically controllable in the set $\mathcal{S}_{\mathrm{D}}$ of states if for any pair of states $\rho_1 \in \mathcal{S}_{\mathrm{D}}$ and $\rho_2 \in \mathcal{S}_{\mathrm{D}}$, there exists a configuration of controls $c(t) \in \mathcal{C}$ and a finite time T, such that the resulting Kraus map $\Phi_{c,T}$ transforms ρ_1 into ρ_2: $\rho_2 = \Phi_{c,T}[\rho_1]$.*

We can also generalize the definition of DSC by considering the asymptotic evolution, $t \to \infty$; the corresponding state is defined as $\rho(\infty) = \lim_{t \to \infty} \Phi_{c,t}[\rho(0)]$ (if the limit exists). The system is asymptotically controllable if the case of $t \to \infty$ is included in definition 6.

Complete DSC will be achieved under the Kraus-map dynamics if $\mathcal{S}_{\mathrm{D}}$ coincides with $\mathcal{S}_{\mathrm{K}} = \mathcal{D}(\mathcal{H})$, i.e., it includes all density operators ρ on the Hilbert space $\mathcal{H}$ of the system. Similar to EOC of closed quantum systems, we can also define Kraus-map controllability (KMC) of open systems:

Definition 7. *An open quantum system with Kraus-map evolution is Kraus-map controllable if the set $\mathcal{C}$ of all available controls generates all maps Φ of the form (2) from the identity map I.*

Definitions 6 and 7 allow us to formulate another corollary of theorem 8:

Corollary 3. *KMC is sufficient for complete DSC of a finite-dimensional open quantum system.*

If any Kraus map can be generated by available controls, then, according to theorem 2 (or, more specifically, corollary 1), any state-to-state transition can be enacted. In general, complete DSC is weaker than KMC, since the former can be achieved even if the set of available controls generates only a subset of all possible Kraus maps. For example, generating all Kraus maps of the form (11) is sufficient for making possible all state-to-state transformations. These maps form only a subset of all Kraus maps, but nevertheless the ability to generate all maps in this subset using some set of controls implies complete DSC.

There can be various dynamical methods to engineer arbitrary finite-time Kraus-map dynamics of open quantum systems. One method relies on the ability to coherently control both the system and environment. Let the system under control be characterized by a Hilbert space $\mathcal{H}_1$ of dimension N. An arbitrary Kraus map $\Phi[\rho] = \sum_{i=1}^{n} K_i \rho K_i^\dagger$ (where n can be chosen such that $n \leqslant N^2$) in the space of states $\mathcal{D}(\mathcal{H}_1)$ can be realized by coupling the system to an ancilla (which serves as an effective environment), characterized by Hilbert space $\mathcal{H}_2$ of

dimension n, and generating a unitary evolution operator U acting in the Hilbert space of the total system $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ as follows [22]. Choose in $\mathcal{H}_2$ a vector $|0\rangle$ and an orthonormal basis $|e_i\rangle, i = 1, \ldots, n$. For any $|\psi\rangle \in \mathcal{H}_1$ let $U(|\psi\rangle \otimes |0\rangle) = \sum_{i=1}^{n} K_i |\psi\rangle \otimes |e_i\rangle$. Then, such operator can be extended to a unitary operator in $\mathcal{H}$ and for any $\rho \in \mathcal{D}(\mathcal{H}_1)$ one has $\Phi[\rho] = \text{Tr}_{\mathcal{H}_2}\{U(\rho \otimes |0\rangle\langle 0|)U^{\dagger}\}$. Therefore the ability to dynamically create, for example via coherent control, an arbitrary unitary evolution of the system and ancilla allows for generating arbitrary Kraus maps of the controlled system. In terms of controllability of finite-dimensional systems, this means that EOC of the system and environment taken together is sufficient for KMC and, according to corollary 3, for complete DSC of the system as well. Since in practice one may not have full control over the environment, Lloyd and Viola [10] proposed another method of Kraus-map engineering, based on a combination of coherent controls and measurements. They have shown that the ability to perform a single simple measurement on the system, together with the ability to apply coherent control to feed back the measurement results, allows for enacting an arbitrary finite-time Kraus-map evolution of the form (2). This procedure determines another set of controls that is sufficient for KMC and, according to corollary 3, for complete DSC of a finite-dimensional open quantum system.

6. Conclusions

This paper establishes and illustrates complete KSC of finite-dimensional open quantum systems with Kraus-map dynamics. The main theoretical result of the paper is that for any target state ρ_f on the Hilbert space of the system, there exists a Kraus map that transforms *all* initial states into ρ_f. The possibility of designing control operations which steer all initial states of the system to a given target state is available due to the use of non-unitary dynamics and is in principle unattainable in the framework of unitary control. Non-unitary controls also allow for transformations between specific pure and mixed states and vice versa. Moreover, there exist different Kraus maps which perform a desired state-to-state transition. For example, the transition between a pair of pure states can be performed via three qualitatively different families of Kraus maps: (i) unitary transformations, (ii) all-to-one non-unitary maps and (iii) one-to-one non-unitary maps.

General definitions of DSC and KMC were introduced for finite-dimensional open quantum systems, leading to the result that KMC is a sufficient condition for complete DSC. Thus, if the available control tools make possible enacting any Kraus map, then any initial state on the Hilbert space of the system can be transformed into any final state. Combining this result with prior findings [10] on Kraus-map engineering determines two specific sets of control operations (i.e., the two methods described in the last paragraph of section 5) which are sufficient for the system to be dynamically controllable in the set of *all* states on the Hilbert space. Moreover, any dynamical control which can produce a Kraus map of theorem 2 by one of these approaches, will be robust to variations in the initial state. Important problems for future research could be to establish the necessary conditions for dynamical controllability of specific open quantum systems and to analyse robustness of the dynamical control to imperfections and environmental effects during the evolution.

Acknowledgments

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